Bailey and William Gosnell, respectively."
(http://www.jstor.org/stable/10.4169/math.mag.85.4.290);
Mathematics Magazine Vol. 85, No. 4 (October 2012), pp. 290-294.

This generalization was also mentioned in the solution submitted by **David Stone and John Hawkins both of Georgia Southern University in Statesboro, GA.** They showed in their solution that all primitive Heronian triangles with sides in arithmetic progression had to have one as the difference in the side lengths. Having a common difference greater than 1 produced a Heronian Triangle, but not a primitive one.

Brian D. Beasley of Presbyterian College in Clinton, SC also stated that this problem is well-known, as noted in [1] (below), and that the sequence $\{n_i\}$ (giving the infinitely many Heronian triangles with side lengths $(n_i - 1, n_i, n_i + 1)$, where $\{n_i\}$ is defined by

$$n_1 = 4,\ n_2 = 14, \text{ and } n_{i+2} = 4n_{i+1} - n_i \text{ for } i \geq 1$$

may also be given in the closed form

$$n_i = (2 + \sqrt{3})^i + (2 - \sqrt{3})^i.$$

Reference:

[1] H. W. Gould, A Triangle with Integral Sides and Area, *The Fibonacci Quarterly*, Vol. 11(1) 1973, 27-39.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego of Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Trey Smith, Angelo State University, San Angelo, TX; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposers.

5471: *Proposed by Arkady Alt, San Jose, CA*

For natural numbers p and n where $n \geq 3$ prove that

$$n^{\frac{1}{n^p}} > (n+p)^{\frac{1}{(n+1)(n+2)(n+3)\cdots(n+p)}}.$$

Solution 1 by Moti Levy, Rehovot, Israel

The function $f(x) = x^{\frac{1}{x}}$ is strictly monotone decreasing for $x \geq 3 > e$, since $f'(x) = x^{\frac{1}{x}} \frac{1}{x^2}(1 - \ln x) < 0, \quad for \quad x > e$. Hence $n + p > n$ implies

$$n^{\frac{1}{n}} > (n+p)^{\frac{1}{(n+p)}}.$$

It follows that

$$\left(n^{\frac{1}{n}}\right)^{\frac{1}{n^{p-1}}} > \left((n+p)^{\frac{1}{(n+p)}}\right)^{\frac{1}{n^{p-1}}},$$

or

$$\left(n^{\frac{1}{n}}\right)^{\frac{1}{n^{p-1}}} = n^{\frac{1}{n^p}} > (n+p)^{\frac{1}{(n+p)n^{p-1}}}.$$

To complete the solution, we note that

$$(n+p)^{\frac{1}{n^{p-1}(n+p)}} > (n+p)^{\frac{1}{(n+1)(n+2)\cdots(n+p)}}.$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

We prove the equivalent inequality

$$\frac{\ln n}{n^p} > \frac{\ln(n+p)}{(n+1)(m+2)\cdots(n+p)}, \qquad (1)$$

by induction on p.

For $x \geq 3$ let $f(x) = \dfrac{\ln x}{x}$. Since $f'(x) = \dfrac{1-\ln x}{x^2} < 0$, so $f(x)$ is strictly decreasing. Hence $f(n) > f(n+1)$ and so (1) is true for $p = 1$. Suppose that (1) is true for $p = k \geq 1$. By the induction assumption, we have

$$\begin{aligned}
\frac{\ln n}{n^{k+1}} &= \frac{1}{n}\left(\frac{\ln n}{n^k}\right) > \frac{\ln(n+k)}{n(n+1)(n+2)\cdots(n+k)} = \\
&= \frac{\ln(n+k+1)}{(n+1)(n+2)\cdots(n+k)(n+k+1)} + \frac{k\ln(n+k)}{n(n+1)(n+2)\cdots(n+k)^2} + \\
&\quad + \frac{1}{(n+1)(n+2)\cdots(n+k)}\left(f(n+k) - f(n+k+1)\right) \\
&> \frac{\ln(n+k+1)}{(n+1)(n+2)\cdots)n+k)(n+k+1)}.
\end{aligned}$$

Thus (1) is true for $p = k+1$ as well and hence true for all positive integers p.

Also solved by Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5472: *Proposed by Francisco Perdomo and Ángel Plaza, both at Universidad Las Palmas de Gran Canaria, Spain*

Let α, β, and γ be the three angles in a non-right triangle. Prove that

$$\frac{1+\sin^2\alpha}{\cos^2\alpha} + \frac{1+\sin^2\beta}{\cos^2\beta} + \frac{1+\sin^2\gamma}{\cos^2\gamma} \geq \frac{1+\sin\alpha\sin\beta}{1-\sin\alpha\sin\beta} + \frac{1+\sin\beta\sin\gamma}{1-\sin\beta\sin\gamma} + \frac{1+\sin\gamma\sin\alpha}{1-\sin\gamma\sin\alpha.}$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We prove more generally that

$$\frac{1+a^2}{1-a^2} + \frac{1+b^2}{1-b^2} + \frac{1+c^2}{1-c^2} \geq \frac{1+ab}{1-ab} + \frac{1+bc}{1-bc} + \frac{1+ca}{1-ca}, \text{ if } 0 \leq a, b, c < 1.$$

The special case follows by putting $a = \sin\alpha, b = \sin\beta, c = \sin\gamma$, with $\alpha + \beta + \gamma = \pi$.